Math 821, Spring 2013, Lecture 8

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1 A Result

Defination 1 *Let $T(x)$ be a formal power series. Define $Supp(T(x))$ to be the set of the indices of nonzero coefficients of $T(x)$. Define $d = \min(Supp(T(x)))$, $g = \gcd(Supp(T(x)) - d)$ ($Supp(T(x)) - d$ means subtract d from each element of the set).*

Eg. For the trees with 0 or 2 children at each vertex,

$$\text{Supp}=\{1,3,5,7,9,\dots\},\ \text{d}=1,\ \text{q}=2.$$

Proposition 1 *Let $T(x) \in xR_{\geq 0}[(x)]$, $E(x,y) \in R_{\geq 0}[(x,y)]$ with $E(0,0) = 0$ and satisfy*

- *$T(x) = E(x, T(x))$ as formal power series,*
- *$0 < \rho < \infty$, $T(\rho) < \infty$ for ρ the radius of convergence of $T(x)$,*
- *$E(x,y)$ has a term of deg ≥ 2 in y,*
- *$\frac{dE(x,y)}{dx} \neq 0$ (so since the coefficients are nonnegative, $\frac{dE(\rho,T(\rho))}{dx} \neq 0$),*
- *$\exists \varepsilon > 0$ such that $E(\rho + \varepsilon, T(\rho) + \varepsilon) < \infty$.*

Then $[x^n]T(x) \sim C\rho^{-n}n^{-\frac{3}{2}}$ *is true for* $n \equiv d \mod \rho$ *with* d, q *as above, and* $[x^n]T(x) = 0$ *otherwise.*

This is a consequence of the 2 results from last time.

The only thing which is outstanding is to check where the singularities are on the circle of convergence.

So to prove the proposition, it suffices to prove

Lemma 1 *Let* E, T, d, q *be as in the propostion. Then the set of singularities of* T *on the circle of convergence is* $\{z : z^q = \rho^q\}$.

Proof. By the def. of d, q, we can write $T(x) = x^d V(x^q)$ for some formal power series $V(x)$.

Thus by Pringsheim's Theorem we have that ρ is a singularity, and every z with $z^q = \rho^q$ is a singularity.

If z is a singularity on the circle of convergence, then the implicit function theorem must fail, so

$$\tfrac{d(y-E(x,y))}{dy}\big|_{x=z,y=T(z)} = 0 \Rightarrow \tfrac{dE(z,T(z))}{dy} = 1,$$

but ρ is also a singularity, so $\frac{dE(z,T(z))}{dy} = \frac{dE(\rho,T(\rho))}{dy}$.

Let $\rho = \gcd\left(Supp[\frac{dE(x,T(x))}{dy}]\right)$.

Again the coefficients are all nonnegative, so the set of singularities on the circle of convergence is contained in $\{z : z^p = \rho^p\}$. So $\{z : z^q = \rho^q\} \subseteq \{z : z^p = \rho^p\}$, $q|p$.

Now just need to check $p \leq q$.

Write out $\frac{dE(x,y)}{dy} = \sum\limits_{n\geq 1} E_n(x)nT(x)^{n-1}$. But also $\frac{dE(x,y)}{dy} = U(x^p)$ for some formal power series U. Pick $a \in Supp(E_n(x))$,

$$Supp(U(x^p)) = Supp(\sum_{n\geq 1} E_n(x)nT(x)^{n-1}) \supseteq a+(n-2)d+Supp(T(x)).$$

Here, $(n-2)d$ is an element of $Supp(T(x))^{n-2}$, the two addition $'+'$ means add this element to all element of $Supp(T(x))$.

Observe for any positive integer m, $\gcd(m + Supp(T(x)))|q$. If $n \in Supp(T(x))$, then let $\gcd(m+Supp(T(x))) = r$, then $r|(m+n)$, $r|(m+d)$, so $r|(n-d)$ which implies $r|q$.

$p = \gcd(Supp(U(x^p))) \leq \gcd(a + (n-2)d + Supp(T(x)))$. Since

$$\gcd(a + (n-2)d + Supp(T(x)))|q,.$$

$p \leq q$. ∎

Now we want to convert combinatorial instructions to $E(x,y)$.

Defination 2 *Given a pair of an operator Θ_Ω and a formal power series $T(x)$, where $\Theta \in \{\mathcal{M}Set, \mathcal{S}eq, \mathcal{U}Cyc, \mathcal{D}Cyc\}$ and $\Omega \subseteq \{1, 2, \ldots\}$. Then define*

$$E^{\Theta,T}(x,y) = \sum_{m\in\Omega} Z\left(G_m, y, T(x^2), \ldots, T(x^m)\right)$$

where

$$G_m = \begin{cases} S_m & \Theta = \mathcal{M}Set \\ E_m & \Theta = \mathcal{S}eq \\ D_m & \Theta = \mathcal{D}Cyc \\ U_m & \Theta = \mathcal{U}Cyc \end{cases} \tag{1}$$

Lemma 2 *Suppose Θ is an operator built out of*

- *power series $E(x,y) \in xR_{\geq 0}[(x,y)]$,*
- *$\mathcal{M}Set_\Omega, \mathcal{S}eq_\Omega, \mathcal{U}Cyc_\Omega, \mathcal{D}Cyc_\Omega$,*
- *$+, \times, \circ$,*

and $T(x)$ is a power series, then $\exists E^{\Theta,T}(x,y)$ such that $\Theta\left(T(x)\right) = E^{\Theta,T}\left(x, T(x)\right)$.

Proof. Define $E^{\Theta,T}$ inductively based on the previous definition. ∎

Defination 3 *An operator as in Lemma 2 is called a composite operator.*

Defination 4 *Say a composite operator Θ is nonlinear if E has a term of deg ≥ 2 in y.*

We saw 2 weeks ago (on A2) that

Lemma 3 *Let Θ be a composite operator with the following properties:*

- $\forall T(x)$, $[x^n]\Theta\left(T(x)\right)$ *depends only on* $[x^j]\Theta\left(T(x)\right)$ *for* $1 \leq j < n$,
- Θ *is bounded in the sense that* $\forall T(x) \in xR_{\geq 0}[(x)]$, $\Theta\left(T(x)\right) \trianglelefteq \sum_{n=1}^{\infty} R^n \left(x + T(x)\right)^n$ *for some* $R > 0$,

then $T(x) = \Theta\left(T(x)\right)$ *has a unique solution, and* $0 < \rho < \infty$, $T(\rho) < \infty$ *where* ρ *is the radius of convergence of* $T(x)$.

Defination 5 *Let* $T(x) \in xR_{\geq 0}[(x)]$ *with radius of convergence* $0 < \rho < \infty$. *A composite operator* Θ *is open for* $T(x)$ *if* $\exists \varepsilon > 0$ *such that* $E^{\Theta,T}\left(\rho + \varepsilon, T(\rho) + \varepsilon\right) < \infty$.

Lemma 4 $T(x) \in xR_{\geq 0}[(x)]$ *with radius* $0 < \rho < \infty$. *Then*

- *a)* $\mathcal{S}eq_{\Omega}$ *is open for* $T(x)$ *iff* Ω *is finite or* $T(\rho) < 1$,
- *b)* $\mathcal{M}Set_{\Omega}$ *is open for* $T(x)$ *iff* $\Omega = \{1\}$ *or* $\rho < 1$,
- *c)* $\mathcal{U}Cyc_{\Omega}$ *and* $\mathcal{D}Cyc_{\Omega}$ *are open for* $T(x)$ *iff*
  - *1)* $\Omega = \{1\}$,
  - *2)* Ω *is finite and* $\rho < 1$,
  - *3)* Ω *is infinite and* $\rho < 1$, $T(\rho) < 1$.

Proof. a) $\mathcal{S}eq_\Omega(T(x)) = \sum\limits_{m\in\Omega} T(x)^m$, so $E(x,y) = \sum\limits_{m\in\Omega} y^m$.

If Ω is finite, this is a polynomial so is open everywhere.

If Ω is infinite and $T(\rho) < 1$, choose ε such that $T(\rho)+\varepsilon < 1$, $\sum\limits_{m\in\Omega}(T(\rho)+\varepsilon)^m$ converges since $\sum\limits_{m=1}^{\infty}(T(\rho)+\varepsilon)^m$ converges. While if $T(\rho) \geq 1$, $\sum\limits_{m\in\Omega}(T(\rho)+\varepsilon)^m \geq \sum\limits_{m\in\Omega} 1 = \infty$.

b) If $\Omega = \{1\}$, $E(x,y) = y$, so open. Otherwise at least one term involving $T(x^k)$ with $k \geq 2$ appears, so if $\rho \geq 1$, $\rho^k \geq \rho$, then diverges. While if $\rho < 1$, then

$$E(x,y) \trianglelefteq e^y exp\,(T(x^m)).$$

Since $\mathcal{M}Set_\Omega \trianglelefteq \mathcal{M}Set$, and when we did the **Pólya** stuff we checked this is open at $(\rho, T(\rho))$.

c) First $\mathcal{D}Cyc_\Omega$.

If $\Omega = \{1\}$, $E(x,y) = y$, so it is ok. Suppose not, then as for $\mathcal{M}Set$, a $T(x^k)$ with $k > 1$ must appear and so $\rho < 1$ is necessary.

Write $E(x,y) = \sum\limits_{m\in\Omega}\frac{1}{m}y^m + \sum\limits_{k\geq 2}\frac{\phi(k)}{k}\sum\limits_{j\cdot k\in\Omega}\frac{1}{j}T(x^k)^j$, where the first part is $A(y)$ and the second part is $B(x)$.

If Ω is finite then $A(y)$ is a polynomial and if $\rho < 1$ then take ε so that $\rho+\varepsilon < 1$ then $B(\rho+\varepsilon) < \infty$.

If Ω is infinite, same argument holds for $B(x)$. The radius of convergence of $A(y)$ is 1, so as in the other parts, $T(\rho) < 1$ is necessary and sufficient for A to be open at $T(\rho)$.

For $\mathcal{U}Cyc$.

$$\mathcal{U}Cyc_\Omega(T(x)) = \frac{1}{2}\mathcal{D}Cyc_\Omega(T(x)) + \frac{1}{4}\sum_{m\in\Omega}\begin{cases} 2T(x)T(x^2)^{\frac{m-1}{2}} & m = 2k+1 \\ T(x)^2T(x^2)^{\frac{m-2}{2}} & m = 2k \end{cases} \tag{2}$$

For E this last part becomes

$$\frac{1}{4}\sum_{m\in\Omega}\begin{cases} yT(x^2)^{\frac{m-1}{2}} & m=2k+1 \\ y^2T(x^2)^{\frac{m-2}{2}}+T(x^2)^{\frac{m}{2}} & m=2k \end{cases} \tag{3}$$

Because of the $\mathcal{D}Cyc$ part, the conditions for $\mathcal{D}Cyc$ are necessary for $\mathcal{U}Cyc$, but the extra part requires only $\rho < 1$, so they are also sufficient. ∎

Lemma 5 *The property of being open at $T(x)$ is closed under $+, \times, \circ$.*

Proof. Just work with the $E^{\Theta,T}$.

Defination 6 *Let $\mathcal{O}$ be the set of composite operators built from*

- $E(x,y) \in Z_{\geq 0}[(x,y)]$, $E(0,0)=0$, E is open where it converges and bounded as in Lemma 3,
- $\mathcal{MS}et_\Omega, \mathcal{S}eq_\Omega, \mathcal{U}Cyc_\Omega, \mathcal{D}Cyc_\Omega$, where for $\mathcal{U}Cyc, \mathcal{D}Cyc$ Ω is finite or $\sum\limits_{m\in\Omega}\frac{1}{m}=\infty$,
- using $+, \times, \circ$.

Lemma 6 *Let $\Theta \in \mathcal{O}$, $T(x) \in xR_{\geq 0}[(x)]$ with radius $0<\rho<1$. Suppose $T(\rho)<\infty$ and $\Theta(T)(\rho)<\infty$. Then Θ is open for T.*

Proof. By induction based on Lemma 4 and Lemma 5. The only problem is $\mathcal{U}Cyc_\Omega, \mathcal{D}Cyc_\Omega$ with Ω infinite.

In this case, we only need to check $T(\rho)<\infty$, but $\sum\limits_{m\in\Omega}\frac{1}{m}=\infty$, so $\Theta(T)(\rho)<\infty$ implies (the $A(y)$ part) that $\sum\limits_{m\in\Omega}\frac{T(\rho)^m}{m}$ converges, so $T(\rho)<1$. ∎

Theorem 1 *Let $\Theta \in \mathcal{O}$, suppose Θ is nonlinear and $[x^n]\Theta(D(x))$ depends only on $[x^j]\Theta(D(x))$ for $1 \leq j < n$. $A(x) \in xR_{\geq 0}[(x)]$ is finite at its radius of convergence. Then $\exists$ a unique $T(x) \in xR_{\geq 0}[(x)]$ such that $T(x)=A(x)+\Theta(T(x))$ and $[x^n]T(x) \sim C\rho^{-n}n^{-\frac{3}{2}}$ for $n \equiv d \mod \rho$ and 0 otherwise (d, q from beginning).*

Proof. Let $\widetilde{\Theta} = A(x) + \Theta(y)$ then $\widetilde{\Theta} \in O$.

By Lemma 3, $\exists$ unique $T(x)$ such that $T(x) = A(x) + \Theta(T(x))$ and $0 < \rho < \infty$, $T(\rho) < \infty$. By integer coefficients, we further get $\rho < 1$. By Lemma 2, we have $E^{\widetilde{\Theta},T}(x,y) = A(x) + E^{\Theta,T}(x,y)$ and $T(x) = E^{\widetilde{\Theta},T}(x,T(x))$.

Now we just need to check the hypothesis of the first proposition of the day.

$A(0) = 0$, and $A(x)$ is finite at its radius of convergence. So $A \neq 0$, $\frac{dA(x)}{dx} \neq 0$, so $\frac{dE^{\widetilde{\Theta},T}(x,y)}{dx} \neq 0$.

And finally we get operators by Lemma 6. ∎

Note:

- The corresponding labelled theorem is true and easier.
- Get interesting examples $T(x) = z + z \times \mathcal{M}Set(\mathcal{D}Cyc_{prime}(\sum 2^n \cdot T^{4n}) + z^8)$.
- Constant C is actually explicit.
- Ref. as same as before, this is the main theorem from the paper.
- If you have an Eg. that doesn't fit in the frame work you'll need to actually integrate. See $F + S$ for a wide variety of useful models.
- What about $\mathcal{P}Set$?

References

[1] Jason P. Bell, Stanley N. Burris, and Karen A. Yeats. Counting rooted trees: the universal law $t(n) \sim C\rho^{-n}n^{-\frac{3}{2}}$. *Electron. J. Combin.*,13(1):Research Paper 63, 64 pp. (electronic), 2006.

[2] Philippe Flajolet and Robert Sedgewick. $IV.2$ $VI.3$. In *Analytic combinatorics*, pages $xiv + 810$. Cambridge University Press, Cambridge, 2009.